Hydrodynamic chains and a classification of their Poisson brackets

Maxim V. Pavlov

Abstract

Necessary and sufficient conditions for an existence of the Poisson brackets significantly simplify in the Liouville coordinates. The corresponding equations can be integrated. Thus, a description of local Hamiltonian structures is a first step in a description of integrable hydrodynamic chains. The concept of M Poisson bracket is introduced. Several new Poisson brackets are presented.

Contents

1 Introduction **2**

2 The general case **4**

3 $M = 0$ **7**

4 $M = 1$ **10**

5 The Miura type and reciprocal transformations **12**

6 Nonlocal Hamiltonian structures **14**

7 Conclusion and outlook **15**

References **16**

keywords: Poisson bracket, Hamiltonian structure, hydrodynamic chain, Liouville coordinates.

MSC: 35L40, 35L65, 37K10;   PACS: 02.30.J, 11.10.E.

1 Introduction

The theory of the hydrodynamic type systems

$$u_t^i = v_j^i(\mathbf{u})u_x^j, \quad i, j = 1, 2, ..., N \tag{1}$$

integrable by the generalized hodograph method (see [24]) starts from the hydrodynamic type systems equipped with the local Hamiltonian structure

$$u_t^i = \{u^i, \bar{\mathbf{h}}\} = [g^{ij}\partial_x - g^{is}\Gamma_{sk}^j u_x^k]\frac{\delta\bar{\mathbf{h}}}{\delta u^j},$$

determined by the Hamiltonian $\bar{\mathbf{h}} = \int h(\mathbf{u})dx$ and the Dubrovin–Novikov bracket (a differential-geometric Poisson bracket of the first order, see [5])

$$\{u^i(x), u^j(x')\} = [g^{ij}\partial_x - g^{is}\Gamma_{sk}^j u_x^k]\delta(x - x'), \qquad i, j = 1, 2, ..., N, \tag{2}$$

where the **flat** metric $g^{ij}(\mathbf{u})$ is symmetric and non-degenerate, $\Gamma_{sk}^j = \frac{1}{2}g^{jm}(\partial_s g_{mk} + \partial_k g_{ms} - \partial_m g_{sk})$ are the Christoffel symbols. Such a local Hamiltonian structure can be written via so-called the *Liouville* coordinates $A^k(\mathbf{u})$, so that the corresponding Poisson bracket is

$$\{A^k(x), A^n(x')\} = [\mathcal{W}^{kn}(\mathbf{A})\partial_x + \partial_x \mathcal{W}^{nk}(\mathbf{A})]\delta(x - x'), \qquad k, n = 1, 2, ..., N. \tag{3}$$

This paper deals with the hydrodynamic chains (cf. (1))

$$U_t^k = \sum_{n=0}^{k+1} V_n^k(\mathbf{U})U_x^n, \qquad k = 0, 1, 2, ..., \tag{4}$$

where all components $V_n^k(\mathbf{U})$ are functions of the field variables U^m and the index m runs from 0 up to $k+1$ (except for possibly *several first* equations, see the end of the next section). This is a very important class of hydrodynamic chains. Plenty known hydrodynamic chains (see [2], [3], [8], [16], [20], [21], [23]) belong to this class. So, the hydrodynamic chains (4) are an extension of hydrodynamic type systems (1) to an infinite component case. Also, these hydrodynamic chains allow the invertible transformations

$$\tilde{U}^0 = \tilde{U}^0(U^0), \quad \tilde{U}^1 = \tilde{U}^1(U^0, U^1), \quad \tilde{U}^2 = \tilde{U}^2(U^0, U^1, U^2), \ ... \tag{5}$$

The Poisson brackets (see (2))

$$\{U^i, U^j\} = [G^{ij}(\mathbf{U})\partial_x + \Gamma_k^{ij}(\mathbf{U})U_x^k]\delta(x - x'), \qquad i, j, k = 1, 2, 3, ... \tag{6}$$

for an infinite component case were considered by I. Dorfman in [4] (N component case was completely investigated by B.A. Dubrovin and S.P. Novikov in [5] for the non-degenerate matrix G^{ij}; its degenerate case was considered by N. Grinberg in [11]).The Jacobi identity yields the set of restrictions (see [4], **Theorem 5.14**)

$$\begin{aligned}
G^{ij} &= G^{ji}, \qquad \partial_k G^{ij} = \Gamma_k^{ij} + \Gamma_k^{ji}, \qquad G^{ik}\Gamma_k^{jn} = G^{jk}\Gamma_k^{in}, \\
0 &= \Gamma_n^{ij}\Gamma_k^{nm} - \Gamma_n^{im}\Gamma_k^{nj} + G^{in}(\partial_n\Gamma_k^{mj} - \partial_k\Gamma_n^{mj}), \\
0 &= (\partial_n\Gamma_k^{ij} - \partial_k\Gamma_n^{ij})\Gamma_p^{nm} + (\partial_n\Gamma_k^{mi} - \partial_k\Gamma_n^{mi})\Gamma_p^{nj} + (\partial_n\Gamma_k^{jm} - \partial_k\Gamma_n^{jm})\Gamma_p^{ni} \\
&\quad + (\partial_n\Gamma_p^{ij} - \partial_p\Gamma_n^{ij})\Gamma_k^{nm} + (\partial_n\Gamma_p^{mi} - \partial_p\Gamma_n^{mi})\Gamma_k^{nj} + (\partial_n\Gamma_p^{jm} - \partial_p\Gamma_n^{jm})\Gamma_k^{ni},
\end{aligned}$$

which in the Liouville coordinates $A^i = A^i(\mathbf{U})$ (see (3) and (5); also [5], [15]) simplify to

$$\begin{aligned}(\mathcal{W}^{ik}+\mathcal{W}^{ki})\partial_k\mathcal{W}^{nj} &= (\mathcal{W}^{jk}+\mathcal{W}^{kj})\partial_k\mathcal{W}^{ni}, \\ \partial_n\mathcal{W}^{ij}\partial_m\mathcal{W}^{kn} &= \partial_n\mathcal{W}^{kj}\partial_m\mathcal{W}^{in}.\end{aligned} \tag{7}$$

In comparison with a finite-component case (hydrodynamic type systems (1)), where elements $\mathcal{W}^{ik}$ depend on *all* field variables u^n $(n = 1, 2, ..., N)$, the elements $\mathcal{W}^{ik}$ in an infinite component case (hydrodynamic chains) depend on a *finite* number of moments A^k only. Thus, the Liouville coordinates A^k in the infinite component case is a natural generalization of the flat coordinates in the finite component case (see [5]).

*The **main** problem in a classification of integrable hydrodynamic chains is the description of the Poisson brackets* (cf. (3))

$$\{A^k(x), A^n(x')\} = [\mathcal{W}^{kn}(\mathbf{A})\partial_x + \partial_x\mathcal{W}^{nk}(\mathbf{A})]\delta(x-x'), \qquad k, n = 1, 2, ...$$

Definition: *The hydrodynamic chain* (4) *is said to be Hamiltonian, if it can be written in the form*

$$A^i_t = [\mathcal{W}^{ij}(\mathbf{A})\partial_x + \partial_x\mathcal{W}^{ji}(\mathbf{A})]\frac{\delta\bar{\mathbf{H}}}{\delta A^j}, \qquad i, j = 0, 1, 2, ..., \tag{8}$$

where the coefficients $\mathcal{W}^{ij}(\mathbf{A})$ *satisfy the Jacobi identity* (7).

The Hamiltonian $\bar{\mathbf{H}} = \int \mathbf{H}dx$ depends on several first moments A^k.

Remark: The above local Hamiltonian structures (8) and corresponding Poisson brackets still are not investigated properly. However, a lot of publications are devoted to their particular cases (see, for instance, [4], [12], [13]), some of them will be described below. In this paper I **present the program** of an investigation of these Poisson brackets.

In this paper we consider the very important case

$$\mathcal{W}^{kn} = \mathcal{W}^{kn}_{(M)}(A^0, A^1, ..., A^{k+n-M}).$$

Corresponding Poisson brackets we call $M-$*brackets*. The simplest sub-cases $\mathcal{W}^{kn} = \mathcal{W}^{kn}_{(0)}(B^0, B^1, ..., B^{k+n})$ and $\mathcal{W}^{kn}_{(1)} = \mathcal{W}^{kn}(A^0, A^1, ..., A^{k+n-1})$ are considered in details below. The first example of such Poisson bracket

$$\{A^k, A^n\} = (kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1})\delta(x-x') \tag{9}$$

was found in [13], the second example is (see [12])

$$\{B^k, B^n\} = [(k+\beta)B^{k+n}\partial_x + (n+\beta)\partial_x B^{k+n}]\delta(x-x'). \tag{10}$$

Infinitely many local Hamiltonian structures for the Benney hydrodynamic chain (see [22]) and for the Kupershmidt hydrodynamic chains (see [21]) are examples of these $M-$brackets. The third example

$$\begin{aligned}\{A^k, A^n\}_2 = {} & [(k+1)A^{k+n}\partial_x + (n+1)\partial_x A^{k+n} + n(k+1)A^{k-1}\partial_x A^{n-1} \\ & + \sum_{m=0}^{n-1}(mA^{n-1-m}\partial_x A^{k-1+m} + (k-n+m)A^{k-1+m}\partial_x A^{n-1-m})]\delta(x-x')\end{aligned}$$

is the second local Hamiltonian structure of the Benney hydrodynamic chain (see [12]), where the moments A^k are no longer the Liouville coordinates. However, this bi-Hamiltonian structure completely determines the Benney hydrodynamic chain together with its commuting flows.

The **main observation** successfully utilized in this paper is that the Jacobi identity (7)

$$\begin{aligned} \sum_{k=0}^{n+j-M}(\mathcal{W}_{(M)}^{ik}+\mathcal{W}_{(M)}^{ki})\partial_k\mathcal{W}_{(M)}^{nj} &= \sum_{k=0}^{n+i-M}(\mathcal{W}_{(M)}^{jk}+\mathcal{W}_{(M)}^{kj})\partial_k\mathcal{W}_{(M)}^{ni}, \\ \sum_{n=0}^{i+j-M}\partial_n\mathcal{W}_{(M)}^{ij}\partial_m\mathcal{W}_{(M)}^{kn} &= \sum_{n=0}^{k+j-M}\partial_n\mathcal{W}_{(M)}^{kj}\partial_m\mathcal{W}_{(M)}^{in}. \end{aligned} \tag{11}$$

is a system of nonlinear PDE's which can be solved completely, because all coefficients $\mathcal{W}_{(M)}^{ij}$ depend on different number of the moments A^k (cf. a similar problem in [20]). Corresponding illustrative examples are given in this paper for $M=0$ and $M=1$.

Remark: Of course, the $M-$bracket is just one among many other examples. For instance, another new Poisson bracket is

$$\{A^k,A^n\}=(A^{k\cdot n}\partial_x+\partial_x A^{k\cdot n})\delta(x-x').$$

All other Poisson brackets (8) and corresponding integrable hydrodynamic chains will be discussed elsewhere.

At the end of this paper local Poisson brackets are generalized on the simplest *nonlocal* case (in N component non-degenerate case known as the nonlocal Hamiltonian structure associated with a metric of constant curvature [9]). As example, a one parametric family of such nonlocal Poisson brackets is presented.

2 The general case

The Poisson brackets determining the hydrodynamic chains (4) written in the Liouville coordinates

$$\{A^k,A^n\}=[\mathcal{W}_{(M)}^{kn}(A^0,A^1,...,A^{k+n-M})\partial_x+\partial_x\mathcal{W}_{(M)}^{nk}(A^0,A^1,...,A^{k+n-M})]\delta(x-x') \tag{12}$$

satisfy the Jacobi identity (11).

The corresponding hydrodynamic chain (4) is

$$A_t^k=\{A^k,\mathbf{H}_{M+1}\}_M=[\mathcal{W}_{(M)}^{kn}(A^0,A^1,...,A^{k+n-M})\partial_x+\partial_x\mathcal{W}_{(M)}^{nk}(A^0,A^1,...,A^{k+n-M})]\frac{\delta\bar{\mathbf{H}}_{M+1}}{\delta A^n},$$

where the Hamiltonian is $\bar{\mathbf{H}}_{M+1}=\int\mathbf{H}_{M+1}(A^0,A^1,A^2,...,A^{M+1})dx$. The Hamiltonian structure is determined by the Hamiltonian, the momentum and the Casimirs. If a hydrodynamic chain has one more conservation law density, then this hydrodynamic chain

is integrable, because every extra conservation law density can be used as an extra Hamiltonian density determining commuting flow (see [12]). Thus, we have several interesting sub-cases:

1. $M=1$. The famous example of such a Poisson bracket is the Kupershmidt–Manin bracket (**9**). This Poisson bracket has the momentum A^1 and the Casimir (annihilator) A^0.

If $M\geqslant 1$, then corresponding Poisson brackets have M Casimirs. Since the Hamiltonian is $\bar{\mathbf{H}}_{M+1}{=}\int\mathbf{H}_{M+1}(A^0,A^1,A^2,...,A^{M+1})dx$, then the momentum can be chosen as $\bar{\mathbf{H}}_M{=}\int A^M dx$. The Casimirs can be chosen as $\bar{\mathbf{H}}_k{=}\int A^k dx$ $(k=0,1,2,...,M-1)$. Then the *auxiliary* (natural) restrictions ("normalization") are

$$\begin{aligned} A^k &= \mathcal{W}_{(M)}^{Mk}(A^0,A^1,...,A^k), \qquad k=0,1,2,..., \\ 0 &= \mathcal{W}_{(M)}^{sk}(A^0,A^1,...,A^{k+s-M}), \qquad 0\leqslant s<M, \qquad k\geqslant M-s. \\ \mathcal{W}_{(M)}^{kn} &= \bar{\mathcal{W}}_{(M)}^{kn}=\mathrm{const}\,, \qquad k=0,1,2,...,M-1, \qquad 0\leqslant n\leqslant M-1-k. \end{aligned} \tag{13}$$

Thus, such a hydrodynamic chain has at least $M+2$ conservation laws (for an arbitrary Hamiltonian density $\mathbf{H}_{M+1}$), where the first N conservation laws of the Casimirs are

$$A_t^k=\partial_x\left(\sum_{n=0}^{M-k-1}(\bar{\mathcal{W}}_{(M)}^{kn}+\bar{\mathcal{W}}_{(M)}^{nk})\frac{\partial\mathbf{H}_{M+1}}{\partial A^n}+\sum_{n=M-k}^{M+1}\mathcal{W}_{(M)}^{nk}\frac{\partial\mathbf{H}_{M+1}}{\partial A^n}\right), \qquad k=0,1,2,...,M-1.$$

The conservation law of the momentum is

$$A_t^M=\partial_x\left(\sum_{n=0}^{M+1}(\mathcal{W}_{(M)}^{nM}+A^n)\frac{\partial\mathbf{H}_{M+1}}{\partial A^n}-\mathbf{H}_{M+1}\right).$$

The conservation law of the energy is

$$\partial_t\mathbf{H}_{M+1}=\partial_x\left[\sum_{k=0}^{M+1}\sum_{n=0}^{M+1}\mathcal{W}_{(M)}^{kn}\frac{\partial\mathbf{H}_{M+1}}{\partial A^k}\frac{\partial\mathbf{H}_{M+1}}{\partial A^n}\right].$$

2. $M=0$. The important example of such a Poisson bracket is the Kupershmidt brackets (**10**). These Poisson brackets have the momentum A^0 only.

3. $M=-1$. These brackets do not have a momentum and annihilators.

4. If $M<-1$, then an investigation of the Poisson brackets and corresponding hydrodynamic chains should start from commuting flows determined by a local Hamiltonian density. For instance, if $M=-2$, then the first nontrivial Hamiltonian hydrodynamic chain (cf. (**4**))

$$A_t^k=\sum_{n=0}^{k+2}V_n^k(\mathbf{A})A_x^n, \qquad k=0,1,2,...$$

is determined by the first local Hamiltonian $\bar{\mathbf{H}}_0{=}\int\mathbf{H}_0(A^0)dx$.

Examples: The Kupershmidt hydrodynamic chains (see [12]) have an infinite set of local Hamiltonian structures determined by $M-$brackets (12), where the first of them are (for the indexes $M=0,1,2$, respectively; see [21])

$$\{C^k,C^n\} = [(\beta k+1)C^{k+n}\partial_x+(\beta n+1)\partial_x C^{k+n}]\delta(x-x'),$$

$$\{B^0,B^0\} = \beta\delta'(x-x'), \qquad \{B^k,B^n\}=[(\beta k+1-\beta)B^{k+n-1}\partial_x+(\beta n+1-\beta)\partial_x B^{k+n-1}]\delta(x-x'),$$

$$\{A^0,A^1\} = \{A^1,A^0\}=\beta\delta'(x-x'), \qquad \{A^1,A^1\}=(\beta-1)(A^0\partial_x+\partial_x A^0)\delta(x-x'),$$

$$\{A^k,A^n\} = [(\beta k+1-2\beta)A^{k+n-2}\partial_x+(\beta n+1-2\beta)\partial_x A^{k+n-2}]\delta(x-x').$$

If the first Poisson bracket is the well-known Kupershmidt bracket ((10), $M=0$), two others ($M=1$ and $M=2$) are new.

The most interesting class of these Poisson brackets is provided by the brackets, whose coefficients $\mathcal{W}^{kn}_{(M)}$ are *polynomials* with respect to the moments A^k. The simplest case is given by the *linear* Poisson brackets determined by

$$\mathcal{W}^{kn}_{(M)}(A^0,A^1,...,A^{k+n-M})=e^{kn}_{(M)}A^{k+n-M},$$

where $e^{kn}_{(M)}$ are some constants satisfying the algebraic system (see (7))

$$(e^{p,k+n-M}_{(M)}+e^{k+n-M,p}_{(M)})e^{nk}_{(M)}=(e^{k,n+p-M}_{(M)}+e^{n+p-M,k}_{(M)})e^{np}_{(M)}, \qquad e^{sp}_{(M)}e^{k,s+p-M}_{(M)}=e^{kp}_{(M)}e^{s,k+p-M}_{(M)},$$

possibly connected with infinite dimensional analogue of the Frobenius algebras (cf. [1]). Let us look for particular solution in the form

$$e^{kn}_{(M)}=(Ak+Bn+C),$$

where A and B are some constants. The substitution of this ansatz in the above algebraic system yields the Kupershmidt brackets (see [12]) determined by an arbitrary value A, but $B=0$. More general linear Poisson brackets

$$\mathcal{W}^{kn}_{(M)}(A^0,A^1,...,A^{k+n-M})=\sum_{s=0}^{k+n-M}e^{kn}_{(M)s}A^s,$$

were considered by I. Dorfman in [4]. For instance,

$$\{A^0,A^0\}=\varepsilon\delta'(x-x'), \qquad \{A^k,A^n\}=\sum_{m=0}^{M}\gamma_m[kA^{m+k+n-1}\partial_x+n\partial_x A^{m+k+n-1}]\delta(x-x'),$$

where ε and γ_k are arbitrary constants.

Higher order homogeneous polynomials create more complicated Poisson brackets, which can be described by algebraic tools. For instance, the second nontrivial case is the *quadratic* Poisson brackets determined by

$$\mathcal{W}_{(M)}^{kn}(A^0,A^1,...,A^{k+n-M})=\frac{1}{2}\sum_{m=0}^{k+n-M}e_{(M)m}^{kn}A^mA^{k+n-m-M},$$

where $e_{(M)m}^{kn}\equiv e_{(M)k+n-m-M}^{kn}$ are some constants, which can be found by direct substitution in the Jacobi identity (see (**11**)). A corresponding system of algebraic relations is very complicated and could be investigated in details elsewhere.

Remark: For every Poisson bracket with respect to highest moment A^{M+1} the Hamiltonian density $\mathbf{H}_{M+1}$ can be *linear* $\mathbf{H}_{M+1}=A^{M+1}+F(A^0,...,A^M)$, *quasilinear* $\mathbf{H}_{M+1}=G(A^0,...,A^M)A^{M+1}+F(A^0,...,A^M)$ and *fully nonlinear*. If $M>0$ just in the *linear* case $\mathbf{H}_{M+1}=A^{M+1}+F(A^0,A^1)$ the hydrodynamic chain has the form (**4**). In general case the corresponding hydrodynamic chain is

$$A_t^k=\sum_{n=0}^{M+1}V_n^k(\mathbf{A})A_x^n,\quad k=0,1,...,M-1,\qquad A_t^k=\sum_{n=0}^{k+1}V_n^k(\mathbf{A})A_x^n,\quad k=M,M+1,...\tag{14}$$

3 $M=0$

We omit the sub-index in all components $\mathcal{W}_{(0)}^{nk}$ of the Poisson bracket (see (**12**), $M=0$)

$$\{B^k,B^n\}=[\mathcal{W}^{kn}(B^0,B^1,...,B^{k+n})\partial_x+\partial_x\mathcal{W}^{nk}(B^0,B^1,...,B^{k+n})]\delta(x-x').$$

The Jacobi identity (**11**)

$$\begin{aligned}\sum_{m=0}^{n+k}(\mathcal{W}^{pm}+\mathcal{W}^{mp})\partial_m\mathcal{W}^{nk} &= \sum_{m=0}^{n+p}(\mathcal{W}^{km}+\mathcal{W}^{mk})\partial_m\mathcal{W}^{np},\\ \sum_{m=0}^{s+p}\partial_m\mathcal{W}^{sp}\partial_n\mathcal{W}^{km} &= \sum_{m=0}^{p+k}\partial_m\mathcal{W}^{kp}\partial_n\mathcal{W}^{sm}.\end{aligned}$$

is a nonlinear PDE system describing a family of the local Poisson brackets connected with the Hamiltonian hydrodynamic chains.

The existence of commuting hydrodynamic chains

$$B_t^k = [\mathcal{W}^{kn}(B^0,B^1,...,B^{k+n})\partial_x+\partial_x\mathcal{W}^{nk}(B^0,B^1,...,B^{k+n})]\frac{\delta\bar{\mathbf{H}}_1}{\delta B^n},\tag{15}$$

$$B_y^k = [\mathcal{W}^{kn}(B^0,B^1,...,B^{k+n})\partial_x+\partial_x\mathcal{W}^{nk}(B^0,B^1,...,B^{k+n})]\frac{\delta\bar{\mathbf{H}}_2}{\delta B^n},\tag{16}$$

where the Hamiltonians are $\bar{\mathbf{H}}_1=\int\mathbf{H}_1(B^0,B^1)dx$ and $\bar{\mathbf{H}}_2=\int\mathbf{H}_2(B^0,B^1,B^2)dx$, implies the existence of a hierarchy of integrable hydrodynamic chains (see (**4**)).

The auxiliary restrictions ("normalization")

$$\mathcal{W}^{0k} \equiv B^k$$

can be obtained by virtue the existence of a conservation law of the momentum $\bar{\mathbf{H}}_0 = \int B^0 dx$. Thus, first two conservation laws are

$$\begin{aligned} B_t^0 &= \partial_x \left(2B^0 \frac{\partial \mathbf{H}_1}{\partial B^0} + (\mathcal{W}^{10} + B^1) \frac{\partial \mathbf{H}_1}{\partial B^1} - \mathbf{H}_1 \right), \\ \partial_t \mathbf{H}_1 &= \partial_x \left[B^0 (\frac{\partial \mathbf{H}_1}{\partial B^0})^2 + (\mathcal{W}^{10} + B^1) \frac{\partial \mathbf{H}_1}{\partial B^0} \frac{\partial \mathbf{H}_1}{\partial B^1} + \mathcal{W}^{11} (\frac{\partial \mathbf{H}_1}{\partial B^0})^2 \right]. \end{aligned}$$

Let us write *several first* nonlinear PDE's from the Jacobi identity (**11**)

$$\begin{aligned} (\mathcal{W}^{10} + B^1)\partial_0 \mathcal{W}^{10} + 2\mathcal{W}^{11}\partial_1 \mathcal{W}^{10} &= 2B^0 \partial_0 \mathcal{W}^{11} + (B^1 + \mathcal{W}^{10})\partial_1 \mathcal{W}^{11} + (B^2 + \mathcal{W}^{20})\partial_2 \mathcal{W}^{11}, \\ \partial_1 \mathcal{W}^{10} \partial_0 \mathcal{W}^{21} &= \partial_1 \mathcal{W}^{20} \partial_0 \mathcal{W}^{11} + \partial_2 \mathcal{W}^{20} \partial_0 \mathcal{W}^{12}, \\ \partial_0 \mathcal{W}^{10} \partial_1 \mathcal{W}^{20} + \partial_1 \mathcal{W}^{10} \partial_1 \mathcal{W}^{21} &= \partial_0 \mathcal{W}^{20} \partial_1 \mathcal{W}^{10} + \partial_1 \mathcal{W}^{20} \partial_1 \mathcal{W}^{11} + \partial_2 \mathcal{W}^{20} \partial_1 \mathcal{W}^{12}, \\ \partial_0 \mathcal{W}^{10} \partial_2 \mathcal{W}^{20} + \partial_1 \mathcal{W}^{10} \partial_2 \mathcal{W}^{21} &= \partial_1 \mathcal{W}^{20} \partial_2 \mathcal{W}^{11} + \partial_2 \mathcal{W}^{20} \partial_2 \mathcal{W}^{12}, \\ \partial_1 \mathcal{W}^{10} \partial_3 \mathcal{W}^{21} &= \partial_2 \mathcal{W}^{20} \partial_3 \mathcal{W}^{12}. \end{aligned}$$

This system involves just four moments B^0, B^1, B^2, B^3. It is enough to find coefficients $\mathcal{W}^{10}$, $\mathcal{W}^{11}$, $\mathcal{W}^{20}$. For instance, $\mathcal{W}^{10}$ is a solution of the Monge–Ampere equation

$$(\mathcal{W}^{10})_{00}(\mathcal{W}^{10})_{11} - [(\mathcal{W}^{10})_{01}]^2 + \varphi'^2(B^0)[(\mathcal{W}^{10})_1]^2 = 0,$$

where $\varphi(B^0)$ is a some function (determined from the compatibility conditions $[(\mathcal{W}^{11})_1]_2 = [(\mathcal{W}^{11})_2]_1$, $[(\mathcal{W}^{11})_1]_0 = [(\mathcal{W}^{11})_0]_1$, $[(\mathcal{W}^{11})_0]_2 = [(\mathcal{W}^{11})_2]_0$, see expressions below).

$$\mathcal{W}^{20} = B^2(\mathcal{W}^{10})_1 + G,$$

where $G(p,q)$ is a solution of the Euler–Darboux–Poisson equation

$$\begin{aligned} G_{pq} &= \frac{\varphi''(B^0)}{4\varphi'^2(B^0)}(G_p - G_q), \qquad \varphi(B^0) = \frac{1}{2}(p-q), \\ q &= \ln[(\mathcal{W}^{10})_1] - \varphi(B^0), \qquad p = \ln[(\mathcal{W}^{10})_1] + \varphi(B^0). \end{aligned}$$

Explicit expressions for all other coefficients (depending on higher moments B^k, $k = 3, 4, 5, ...$) can be found recursively in *complete* differentials. For instance, the equation

$$(\mathcal{W}^{11})_2 = \frac{2[(\mathcal{W}^{10})_1]^2 (B^0 (\mathcal{W}^{10})_{00} + (\mathcal{W}^{10} + B^1)(\mathcal{W}^{10})_{10} + \mathcal{W}^{11}(\mathcal{W}^{10})_{11})}{MB^2 + N},$$

where

$$M = [1+(\mathcal{W}^{10})_1](\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11} + 2B^0[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{110} - (\mathcal{W}^{10})_{10}(\mathcal{W}^{10})_{11}]$$
$$+(\mathcal{W}^{10}+B^1)[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{111} - ((\mathcal{W}^{10})_{11})^2],$$
$$N = G(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11} + 2B^0[G_{10}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{10}]$$
$$+[(\mathcal{W}^{10})_1+B^1][G_{11}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{11}],$$

can be solved up to a some function of B^0 and B^1, which can be found in complete differentials by a substitution into other derivatives

$$(\mathcal{W}^{11})_1 = (\mathcal{W}^{11})_2\frac{[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{111} - ((\mathcal{W}^{10})_{11})^2]B^2 + G_{11}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{11}}{(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11}} + K,$$
$$(\mathcal{W}^{11})_0 = (\mathcal{W}^{11})_2\frac{[(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{110} - (\mathcal{W}^{10})_{10}(\mathcal{W}^{10})_{11}]B^2 + G_{10}(\mathcal{W}^{10})_1 - G_1(\mathcal{W}^{10})_{10}}{(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{11}} - L,$$

where

$$K = \frac{(\mathcal{W}^{10})_0(\mathcal{W}^{10})_{11} - 2(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{10}}{(\mathcal{W}^{10})_{11}}, \qquad L = \frac{(\mathcal{W}^{10})_1(\mathcal{W}^{10})_{00}}{(\mathcal{W}^{10})_{11}}.$$

Let us write the first two equations from the (**15**) and the first equation from (**16**)

$$B_t^0 = (B^0\partial_x + \partial_x B^0)\frac{\delta\bar{\mathbf{H}}_1}{\delta B^0} + [B^1\partial_x + \partial_x\mathcal{W}^{10}(B^0,B^1)]\frac{\delta\bar{\mathbf{H}}_1}{\delta B^1},$$
$$B_t^1 = [\mathcal{W}^{10}(B^0,B^1)\partial_x + \partial_x B^1]\frac{\delta\bar{\mathbf{H}}_1}{\delta B^0} + [\mathcal{W}^{11}(B^0,B^1,B^2)\partial_x + \partial_x\mathcal{W}^{11}(B^0,B^1,B^2)]\frac{\delta\bar{\mathbf{H}}_1}{\delta B^1},$$
$$B_y^0 = (B^0\partial_x + \partial_x B^0)\frac{\delta\bar{\mathbf{H}}_2}{\delta B^0} + [B^1\partial_x + \partial_x\mathcal{W}^{10}(B^0,B^1)]\frac{\delta\bar{\mathbf{H}}_2}{\delta B^1} + [B^2\partial_x + \partial_x\mathcal{W}^{20}(B^0,B^1,B^2)]\frac{\delta\bar{\mathbf{H}}_2}{\delta B^2}.$$

These three equations can be written in the conservative form

$$\partial_t\mathbf{H}_0 = \partial_x F_{00}(\mathbf{H}_0,\mathbf{H}_1), \qquad \partial_t\mathbf{H}_1 = \partial_x F_{01}(\mathbf{H}_0,\mathbf{H}_1,\mathbf{H}_2), \qquad \partial_y\mathbf{H}_0 = \partial_x F_{10}(\mathbf{H}_0,\mathbf{H}_1,\mathbf{H}_2).$$

Let us introduce the potential function z, where $\mathbf{H}_0 = z_x$, $F_{00}(\mathbf{H}_0,\mathbf{H}_1) = z_t$, $F_{10}(\mathbf{H}_0,\mathbf{H}_1,\mathbf{H}_2) = z_y$. Then one can substitute $\mathbf{H}_1(z_x,z_t)$ and $\mathbf{H}_2(z_x,z_t,z_y)$ into the second equation

$$\partial_t\mathbf{H}_1(z_x,z_t) = \partial_x F_{01}(\mathbf{H}_0,\mathbf{H}_1(z_x,z_t),\mathbf{H}_2(z_x,z_t,z_y)).$$

This 2+1 quasilinear equation of the second order is integrable by the method of hydrodynamic reductions [7] (or by introducing pseudopotentials; see again [7] and [25]). The Hamiltonian structure of this equation is unknown, while the Hamiltonian structure of corresponding hydrodynamic chains (**15**) and (**16**) is defined.

4 $M=1$

We omit the sub-index in all components $\mathcal{W}^{nk}_{(1)}$ of the Poisson bracket (see (**12**), $M=1$)

$$\{A^k,A^n\}=[\mathcal{W}^{kn}(A^0,A^1,...,A^{k+n-1})\partial_x+\partial_x\mathcal{W}^{nk}(A^0,A^1,...,A^{k+n-1})]\delta(x-x').$$

The Jacobi identity (**11**) is

$$\begin{aligned}\sum_{m=0}^{n+k-1}(\mathcal{W}^{pm}+\mathcal{W}^{mp})\partial_m\mathcal{W}^{nk} &= \sum_{m=0}^{n+p-1}(\mathcal{W}^{km}+\mathcal{W}^{mk})\partial_m\mathcal{W}^{np},\\ \sum_{m=0}^{s+p-1}\partial_m\mathcal{W}^{sp}\partial_n\mathcal{W}^{km} &= \sum_{m=0}^{p+k-1}\partial_m\mathcal{W}^{kp}\partial_n\mathcal{W}^{sm}.\end{aligned}$$

Thus, the above system of nonlinear PDE's describes a family of the local Poisson brackets connected with the Hamiltonian hydrodynamic chains.

The existence of commuting hydrodynamic chains

$$\begin{aligned}A^k_t &= [\mathcal{W}^{kn}(A^0,A^1,...,A^{k+n-1})\partial_x+\partial_x\mathcal{W}^{nk}(A^0,A^1,...,A^{k+n-1})]\frac{\delta\bar{\mathbf{H}}_2}{\delta A^n}, \qquad (17)\\ A^k_y &= [\mathcal{W}^{kn}(A^0,A^1,...,A^{k+n-1})\partial_x+\partial_x\mathcal{W}^{nk}(A^0,A^1,...,A^{k+n-1})]\frac{\delta\bar{\mathbf{H}}_3}{\delta A^n},\end{aligned}$$

where the Hamiltonians are $\bar{\mathbf{H}}_2=\int\mathbf{H}_2(A^0,A^1,A^2)dx$ and $\bar{\mathbf{H}}_3=\int\mathbf{H}_3(A^0,A^1,A^2,A^3)dx$, implies the existence of a hierarchy of integrable hydrodynamic chains.

The *auxiliary* restrictions ("normalization")

$$\mathcal{W}^{0k}(A^0,A^1,...,A^{k-1})\equiv\bar{\mathcal{W}}^{00}\delta^{0k}, \qquad \mathcal{W}^{1k}(A^0,A^1,...,A^k)\equiv A^k,$$

where $\bar{\mathcal{W}}^{00}=\text{const}$ (and δ^{ik} is the Kronecker symbol), can be obtained by virtue of an existence of conservation laws of the Casimir $\bar{\mathbf{H}}_0=\int A^0dx$ and the momentum $\bar{\mathbf{H}}_1=\int A^1dx$. Then the first three conservation laws are

$$\begin{aligned}A^0_t &= \partial_x\left(2\bar{\mathcal{W}}^{00}\frac{\partial\mathbf{H}_2}{\partial A^0}+A^0\frac{\partial\mathbf{H}_2}{\partial A^1}+\mathcal{W}^{20}\frac{\partial\mathbf{H}_2}{\partial A^2}\right),\\ A^1_t &= \partial_x\left(A^0\frac{\partial\mathbf{H}_2}{\partial A^0}+2A^1\frac{\partial\mathbf{H}_2}{\partial A^1}+(\mathcal{W}^{21}+A^2)\frac{\partial\mathbf{H}_2}{\partial A^2}-\mathbf{H}_2\right),\\ \partial_t\mathbf{H}_2 &= \partial_x[\varepsilon(\frac{\partial\mathbf{H}_2}{\partial A^0})^2+A^0\frac{\partial\mathbf{H}_2}{\partial A^0}\frac{\partial\mathbf{H}_2}{\partial A^1}+\mathcal{W}^{20}\frac{\partial\mathbf{H}_2}{\partial A^0}\frac{\partial\mathbf{H}_2}{\partial A^2}\\ &\quad +A^1(\frac{\partial\mathbf{H}_2}{\partial A^1})^2+(\mathcal{W}^{21}+A^2)\frac{\partial\mathbf{H}_2}{\partial A^1}\frac{\partial\mathbf{H}_2}{\partial A^2}+\mathcal{W}^{22}(\frac{\partial\mathbf{H}_2}{\partial A^2})^2],\end{aligned}$$

where $\varepsilon\equiv\bar{\mathcal{W}}^{00}$.

Let us write *several first* nonlinear PDE's from the Jacobi identity (**11**)

$$
\begin{aligned}
[2\varepsilon\partial_0 + A^0\partial_1 + \mathcal{W}^{20}\partial_2]\mathcal{W}^{21} &= [A^0\partial_0 + 2A^1\partial_1]\mathcal{W}^{20} \\
\partial_1\mathcal{W}^{20}\partial_0\mathcal{W}^{31} &= \partial_1\mathcal{W}^{30}\partial_0\mathcal{W}^{21} + \partial_2\mathcal{W}^{30}\partial_0\mathcal{W}^{22}, \\
\partial_0\mathcal{W}^{30}\partial_1\mathcal{W}^{20} + \partial_1\mathcal{W}^{30}\partial_1\mathcal{W}^{21} + \partial_2\mathcal{W}^{30}\partial_1\mathcal{W}^{22} &= \partial_0\mathcal{W}^{20}\partial_1\mathcal{W}^{30} + \partial_1\mathcal{W}^{20}\partial_1\mathcal{W}^{31}, \\
\partial_0\mathcal{W}^{20}\partial_2\mathcal{W}^{30} + \partial_1\mathcal{W}^{20}\partial_2\mathcal{W}^{31} &= \partial_1\mathcal{W}^{30}\partial_2\mathcal{W}^{21} + \partial_2\mathcal{W}^{30}\partial_2\mathcal{W}^{22}, \\
\partial_1\mathcal{W}^{20}\partial_3\mathcal{W}^{31} &= \partial_2\mathcal{W}^{30}\partial_3\mathcal{W}^{22}.
\end{aligned}
$$

This system involves just four moments A^0, A^1, A^2, A^3. It is enough to find coefficients $\mathcal{W}^{20}$, $\mathcal{W}^{21}$, $\mathcal{W}^{30}$. For instance, $\mathcal{W}^{20}$ is a solution of the Monge–Ampere equation

$$
(\mathcal{W}^{20})_{00}(\mathcal{W}^{20})_{11} - [(\mathcal{W}^{20})_{01}]^2 + \varphi'^2(A^0)[(\mathcal{W}^{20})_1]^2 = 0, \tag{18}
$$

where $\varphi(A^0)$ is a some function (determined from the compatibility conditions $[(\mathcal{W}^{21})_1]_2 = [(\mathcal{W}^{21})_2]_1,\ [(\mathcal{W}^{21})_1]_0 = [(\mathcal{W}^{21})_0]_1,\ [(\mathcal{W}^{21})_0]_2 = [(\mathcal{W}^{21})_2]_0$, see these expressions below).

$$
\mathcal{W}^{30} = A^2(\mathcal{W}^{20})_1 + G,
$$

where $G(p,q)$ is a solution of the Euler–Darboux–Poisson equation

$$
\begin{aligned}
G_{pq} &= \frac{\varphi''(A^0)}{4\varphi'^2(A^0)}(G_p - G_q), \qquad \varphi(A^0) = \frac{1}{2}(p-q), \\
q &= \ln[(\mathcal{W}^{20})_1] - \varphi(A^0), \qquad p = \ln[(\mathcal{W}^{20})_1] + \varphi(A^0).
\end{aligned}
$$

Explicit expressions for all other coefficients (depending on higher moments A^k, $k = 3, 4, 5, \ldots$) can be found recursively in *complete* differentials. For instance,

$$
d\mathcal{W}^{21} = (\mathcal{W}^{21})_0 dA^0 + (\mathcal{W}^{21})_1 dA^1 + (\mathcal{W}^{21})_2 dA^2,
$$

where

$$
\begin{aligned}
(\mathcal{W}^{21})_2 &= \frac{2[A^1(\mathcal{W}^{20})_{11} + A^0(\mathcal{W}^{20})_{10} + \varepsilon(\mathcal{W}^{20})_{00}]}{2\varepsilon\left(\frac{(\mathcal{W}^{20})_{11}A^2+G_1}{(\mathcal{W}^{20})_1}\right)_0 + A^0\left(\frac{(\mathcal{W}^{20})_{11}A^2+G_1}{(\mathcal{W}^{20})_1}\right)_1 + \frac{(\mathcal{W}^{20})(\mathcal{W}^{20})_{11}}{(\mathcal{W}^{20})_1}} \\
(\mathcal{W}^{21})_1 &= \frac{(\mathcal{W}^{20})_1}{(\mathcal{W}^{20})_{11}}\left(\frac{(\mathcal{W}^{20})_{11}A^2 + G_1}{(\mathcal{W}^{20})_1}\right)_1(\mathcal{W}^{21})_2 + (\mathcal{W}^{20})_0 - 2\frac{(\mathcal{W}^{20})_1(\mathcal{W}^{20})_{10}}{(\mathcal{W}^{20})_{11}} \\
(\mathcal{W}^{21})_0 &= \frac{(\mathcal{W}^{20})_1}{(\mathcal{W}^{20})_{11}}\left(\frac{(\mathcal{W}^{20})_{11}A^2 + G_1}{(\mathcal{W}^{20})_1}\right)_0(\mathcal{W}^{21})_2 - \frac{(\mathcal{W}^{20})_1(\mathcal{W}^{20})_{00}}{(\mathcal{W}^{20})_{11}}.
\end{aligned}
$$

Remark: The coincidence of the coefficients $\mathcal{W}^{20}$ from this section and $\mathcal{W}^{10}$ from the previous section is easy to understand if one takes into account that if the Hamiltonian

density $\mathbf{H}_2$ is a function of the moments A^1 and A^2 only, then the corresponding hydrodynamic chain has the Hamiltonian structure coinciding (by the "shift" $A^k \rightarrow B^{k-1}$, $k = 1, 2, ...$) with the Hamiltonian structure presented in the previous section.

Remark: The Monge–Ampere equation (18) was derived in [8]. In this paper solutions of this equation classify the integrable hydrodynamic chains

$$\partial_t \mathbf{H}_0 = \partial_x F_1(\mathbf{H}_0, \mathbf{H}_1), \quad \partial_t \mathbf{H}_1 = \partial_x F_2(\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2), \quad \partial_t \mathbf{H}_2 = \partial_x F_3(\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3), \ ...$$

These hydrodynamic chains *can coincide* with the integrable hydrodynamic chains (17) determined by the Hamiltonian density $\mathbf{H}_2 = A^2 + Q(A^0, A^1)$.

All other Hamiltonian hydrodynamic chains (8) can be investigated in the same way using the Jacobi identity (7).

5 The Miura type and reciprocal transformations

Suppose two Hamiltonian hydrodynamic chains

$$B_t^k = \sum_{n=0}^{N_k} F_n^k(\mathbf{B}) B_x^n, \qquad A_t^k = \sum_{m=0}^{M_k} G_n^k(\mathbf{A}) A_x^n, \qquad k = 1, 2, ...,$$

where N_k and M_k are some integers, are related by the Miura type transformations

$$A^k = A^k(B^0, B^1, ..., B^{k+1}), \qquad k = 0, 1, 2, ...$$

Theorem: *The corresponding Poisson brackets* (*see* (6) *and* (12))

$$\{B^k, B^n\} = [G_{(L)}^{kn}(B^0, B^1, ..., B^{k+n-L})\partial_x + \Gamma_{(L)m}^{kn}(\mathbf{B}) B_x^m]\delta(x - x'), \tag{19}$$

$$\{A^k, A^n\} = [G_{(M)}^{kn}(A^0, A^1, ..., A^{k+n-M})\partial_x + \Gamma_{(M)m}^{kn}(\mathbf{A}) A_x^m]\delta(x - x') \tag{20}$$

are related by the above Miura type transformations if $L = M + 1$.

Proof: It is easy to check by taking into account the *highest order* dependence on the moments B^n only.

$$\{A^k, A^n\} = \sum_{i=0}^{k+1}\sum_{j=0}^{n+1} \frac{\partial A^k}{\partial B^i}\{B^i, B^j\}\frac{\partial A^n}{\partial B^j}\delta(x - x') \sim \frac{\partial A^k}{\partial B^{k+1}}\{B^{k+1}, B^{n+1}\}\frac{\partial A^n}{\partial B^{n+1}}\delta(x - x').$$

Thus, $G_{(M)}^{kn}(A^0, A^1, ..., A^{k+n-M}) \sim G_{(L)}^{k+1,n+1}(B^0, B^1, ..., B^{k+n+2-L})$. So, indeed, $L = M+1$.

Remark: Suppose that both coordinate systems are the Liouville coordinates. Then, one obtains the linear system describing such Miura type transformations

$$\partial_m \left[\frac{\partial A^k}{\partial B^i}[(\mathcal{W}_{(L)}^{ij} + \mathcal{W}_{(L)}^{ji})\frac{\partial^2 A^n}{\partial B^j \partial B^s} + \mathcal{W}_{(L)s}^{ji}\frac{\partial A^n}{\partial B^j}]\right] = \partial_s \left[\frac{\partial A^k}{\partial B^i}[(\mathcal{W}_{(L)}^{ij} + \mathcal{W}_{(L)}^{ji})\frac{\partial^2 A^n}{\partial B^j \partial B^m} + \mathcal{W}_{(L)m}^{ji}\frac{\partial A^n}{\partial B^j}]\right].$$

If the first Hamiltonian structure written in the Liouville coordinates (19) and the Miura type transformations (12) are given, then the second Hamiltonian structure (20) can be reconstructed in complete differentials, where

$$\frac{\partial \mathcal{W}_{(M)}^{nk}(\mathbf{A})}{\partial B^s} = \frac{\partial A^k}{\partial B^i}[(\mathcal{W}_{(L)}^{ij} + \mathcal{W}_{(L)}^{ji})\frac{\partial^2 A^n}{\partial B^j \partial B^s} + \mathcal{W}_{(L)s}^{ji}\frac{\partial A^n}{\partial B^j}].$$

Example: The Benney hydrodynamic chain (see (9), [2] and [13])

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0 = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}]\frac{\partial \mathbf{H}_2}{\partial A^n}, \qquad k = 0, 1, 2, ...,$$

where the Hamiltonian density $\mathbf{H}_2 = A^2 + (A^0)^2$, is connected with the modified Benney hydrodynamic chain (see [23], the particular case of the Kupershmidt hydrodynamic chains [12])

$$B_t^k = B_x^{k+1} + B^0 B_x^k + (k+2)B^k B_x^0 = [(k+1)B^{k+n}\partial_x + (n+1)\partial_x B^{k+n}]\frac{\partial \mathbf{H}_0}{\partial B^n},$$

where the Hamiltonian density $\mathbf{H}_0 = B^1 + (B^0)^2$, by the Miura type transformations (see [22])

$$A^0 = B^1 + (B^0)^2, \;\; A^1 = B^2 + 3B^0B^1 + 2(B^0)^3, ...$$

In this case $L = 0$ and $M = -1$. Thus, the Benney moment chain has the second local Hamiltonian structure determined by the Poisson bracket (20), whose coefficients can be found by using the above Miura type transformations.

Remark: In the theory of dispersive integrable systems the Miura transformation is a tool to reduce the Hamiltonian structure to canonical form "d/dx" (the infinitely many component analogue of the Darboux theorem, see [19]). In the theory of the Poisson brackets associated with hydrodynamic chains an application of the Miura type transformation reduces the dependence on a number of the moments A^k. It means that the following diagram exists (see (6) **and** (12))

$$\begin{array}{rcl} \{A^k, A^n\} & = & [G^{kn}_{(-M)}(A^0, ..., A^{k+n+M}) + ...]\delta(x-x'), \\ & \downarrow & \\ \{B^k, B^n\} & = & [G^{kn}_{(1-M)}(B^0, ..., B^{k+n+M-1}) + ...]\delta(x-x'), \\ & \downarrow & \\ \{C^k, C^n\} & = & [G^{kn}_{(2-M)}(C^0, ..., C^{k+n+M-2}) + ...]\delta(x-x'), \\ & \downarrow & \\ & & ..... \end{array}$$

Conjecture: Possibly, the number of such Miura type transformations is *infinite*, and any local Hamiltonian structure with the index $-M$ can be reduced to local Hamiltonian structure with an arbitrary index N, where the triangular block $\bar{\mathcal{W}}^{kn}_{(N)} = \text{const}$ increases (see (13), if N increases) in complete accordance with the Darboux theorem.

Let us apply the reciprocal transformation

$$dz = \mathbf{F}(A^0, A^1, ...)dx + \mathbf{G}(A^0, A^1, ...)dt, \qquad dy = dt$$

to the hydrodynamic chain (14)

$$A_t^k = \sum_{n=0}^{M+1} V_n^k(\mathbf{A})A_x^n, \quad k = 0, 1, ..., M-1, \quad A_t^k = \sum_{n=0}^{k+1} V_n^k(\mathbf{A})A_x^n, \quad k = M, M+1, ...$$

If the conservation law density $\mathbf{F}(A^0, A^1, ...)$ is a linear function (with special choice of constants, see [6]) of M Casimirs A^k and the momentum A^M, then new hydrodynamic chain

$$A_y^k = \sum_{n=0}^{M+1} (\mathbf{F}V_n^k(\mathbf{A}) - \mathbf{G}\delta_n^k) A_z^n, \;\; k = 0, 1, ..., M-1, \quad A_y^k = \sum_{n=0}^{k+1} (\mathbf{F}V_n^k(\mathbf{A}) - \mathbf{G}\delta_n^k) A_z^n, \;\; k = M, M+1, ...$$

preserves the *local* Hamiltonian structure (see (6) and (12)), i.e.

$$\begin{array}{rcl} \{A^k(x), A^n(x')\} & = & [G^{kn}_{(M)}(A^0, ..., A^{k+n-M}) + ...]\delta(x - x'), \\ & \downarrow & \\ \{A^k(z), A^n(z')\} & = & [\tilde{G}^{kn}_{(M)}(A^0, ..., A^{k+n-M}) + ...]\delta(z - z'). \end{array}$$

Possibly, more general (generalized) reciprocal transformations (see [6] and [10]) preserving local Hamiltonian structures for integrable hydrodynamic chains can be found.

6 Nonlocal Hamiltonian structures

The nonlocal Poisson brackets (cf. (2))

$$\{U^i, U^j\} = [G^{ij}(\mathbf{U})\partial_x + \Gamma_k^{ij}(\mathbf{U})U_x^k + \varepsilon U_x^i \partial_x^{-1} U_x^j]\delta(x - x'), \qquad i, j, k = 1, 2, 3, ... \tag{21}$$

for N component case were completely investigated by E.V. Ferapontov and O.I. Mokhov in [9] for the non-degenerate matrix G^{ij}; its degenerate case was considered by O.I. Mokhov in [17]).The Jacobi identity yields the set of restrictions

$$G^{ij} = G^{ji}, \qquad \partial_k G^{ij} = \Gamma_k^{ij} + \Gamma_k^{ji}, \qquad G^{ik}\Gamma_k^{jn} = G^{jk}\Gamma_k^{in},$$

$$\varepsilon(G^{im}\delta_k^j - G^{ij}\delta_k^m) = \Gamma_n^{ij}\Gamma_k^{nm} - \Gamma_n^{im}\Gamma_k^{nj} + G^{in}(\partial_n\Gamma_k^{mj} - \partial_k\Gamma_n^{mj}),$$

$$-\varepsilon[(\Gamma_k^{ij} - \Gamma_k^{ji})\delta_p^m + (\Gamma_k^{mi} - \Gamma_k^{im})\delta_p^j + (\Gamma_k^{jm} - \Gamma_k^{mj})\delta_p^i$$

$$+(\Gamma_p^{ij} - \Gamma_p^{ji})\delta_k^m + (\Gamma_p^{mi} - \Gamma_p^{im})\delta_k^j + (\Gamma_p^{jm} - \Gamma_p^{mj})\delta_k^i]$$

$$=$$

$$(\partial_n\Gamma_k^{ij} - \partial_k\Gamma_n^{ij})\Gamma_p^{nm} + (\partial_n\Gamma_k^{mi} - \partial_k\Gamma_n^{mi})\Gamma_p^{nj} + (\partial_n\Gamma_k^{jm} - \partial_k\Gamma_n^{jm})\Gamma_p^{ni}$$

$$+(\partial_n\Gamma_p^{ij} - \partial_p\Gamma_n^{ij})\Gamma_k^{nm} + (\partial_n\Gamma_p^{mi} - \partial_p\Gamma_n^{mi})\Gamma_k^{nj} + (\partial_n\Gamma_p^{jm} - \partial_p\Gamma_n^{jm})\Gamma_k^{ni},$$

which in the Liouville coordinates $A^i = A^i(\mathbf{U})$ determined by the conditions (see [15], [17])

$$G^{ij} = \mathcal{W}^{ij} + \mathcal{W}^{ji} - \varepsilon A^i A^j, \qquad \Gamma_k^{ij} = \partial_k \mathcal{W}^{ji} - \varepsilon\delta_k^i A^j,$$

simplify to (cf. (7))

$$
\begin{aligned}
(\mathcal{W}^{ik}+\mathcal{W}^{ki}-\varepsilon A^{i}A^{k})\partial_{k}\mathcal{W}^{nj} &= (\mathcal{W}^{jk}+\mathcal{W}^{kj}-\varepsilon A^{j}A^{k})\partial_{k}\mathcal{W}^{ni}, \\
\partial_{n}\mathcal{W}^{ij}\partial_{m}\mathcal{W}^{kn} &= \partial_{n}\mathcal{W}^{kj}\partial_{m}\mathcal{W}^{in}.
\end{aligned} \tag{22}
$$

Thus, the nonlocal Poisson bracket in the Liouville coordinates has the form (the numeration of the moments A^k runs from 0 up to infinity)

$$\{A^i,A^j\}=[\mathcal{W}^{ij}\partial_x+\partial_x\mathcal{W}^{ji}-\varepsilon A^j\partial_x A^i+\varepsilon A^i_x\partial_x^{-1}A^j_x]\delta(x-x'),\qquad i,j=0,1,...$$

Similar classification of these nonlocal Poisson brackets (see (22)) will be made elsewhere.

The first example of such a nonlocal Poisson bracket (21) is (see [21])

$$
\begin{aligned}
\{U^k,U^n\} &= [\beta(\beta k+\beta+1)U^{k+n+1}\partial_x+\beta(\beta n+\beta+1)\partial_x U^{k+n+1} \\
&+(\beta k+\beta+2)(\beta n+\beta+2)U^kU^n\partial_x+(\beta k+\beta+2)(\beta n+\beta+1)U^k(U^n)_x \\
&+(\beta n+\beta+2)U^n(U^k)_x-(U^k)_x\partial_x^{-1}(U^n)_x]\delta(x-x').
\end{aligned}
$$

Remark: A more general Poisson brackets have been introduced by E.V. Ferapontov (see [6]) for N component non-degenerate case. A similar extension as in the above case one can develop for the infinite component case. Then the Poisson bracket in the Liouville coordinates is (see [18])

$$
\begin{aligned}
\{U^k,U^n\} &= [(\Phi^{kn}+\Phi^{nk}-\varepsilon_{pq}V^{(p)k}V^{(q)n})\partial_x+(\partial_m\Phi^{nk}-\varepsilon_{pq}V^{(q)n}\partial_m V^{(p)k})A^m_x \\
&+\varepsilon_{pq}\partial_m V^{(p)k}A^m_x\partial_x^{-1}\partial_s V^{(q)n}A^s_x]\delta(x-x'),
\end{aligned}
$$

where Φ^{kn} and $V^{(p)k}$ are functions of the moments A^m, the constant LxL matrix ε_{pq} is symmetric and non-degenerated (L is "co-dimension" of the pseudo-Riemannian space).

7 Conclusion and outlook

The **main claim** of this paper is that the Poisson brackets

$$\{A^k,A^n\}=[\mathcal{W}^{kn}_{(M)}(A^0,...,A^{k+n-M})\partial_x+\partial_x\mathcal{W}^{nk}_{(M)}(A^0,...,A^{k+n-M})]\delta(x-x'),\quad k,n=0,1,2,...$$

can be classified completely by virtue of the Jacobi identity (7), which is the overdetermined system of infinitely many nonlinear PDE's. The **main observation** is that the solution of the several first nonlinear PDE's allows one to compute all other coefficients recursively in complete differentials.

Nevertheless, the problem of complete description of these Poisson brackets is solvable, because all integrable hydrodynamic chains (4) are already found (see [8]). Thus, one

should just identify this classification and hydrodynamic chains generated by the Poisson brackets described above.

Conjecture: *Moreover, we believe that* ***any integrable hydrodynamic chain***

$$A_t^k = \sum_{n=0}^{N_k} V_n^k(\mathbf{A}) A_x^n, \qquad k = 0, 1, 2, ...,$$

where set of functions $V_n^k(\mathbf{A})$ *depends on the moments* A^m $(m = 0, 1, ..., M_k;\ M_k$ *and* N_k *are arbitrary integers), possesses* ***infinitely many local Hamiltonian structures*** (8).

Let us consider a bi-Hamiltonian structure (see, for instance, (9) and below) determined by the couple of $M-$brackets

$$A_t^k = [\mathcal{W}_{(M)}^{kn}\partial_x + \partial_x \mathcal{W}_{(M)}^{nk}] \frac{\delta \mathbf{H}_{M+1}}{\delta A^n} = [G_{(M+1)}^{kn} + \Gamma_{(M+1)s}^{kn} A_x^s] \frac{\delta \mathbf{H}_{M+2}}{\delta A^n}$$

or, for instance,

$$A_t^k = [\mathcal{W}_{(M)}^{kn}\partial_x + \partial_x \mathcal{W}_{(M)}^{nk}] \frac{\delta \mathbf{H}_{M+1}}{\delta A^n} = [\tilde{G}_{(M-1)}^{kn} + \tilde{\Gamma}_{(M-1)s}^{kn} A_x^s] \frac{\delta \mathbf{H}_{M}}{\delta A^n}.$$

Suppose, that all $M-$brackets are classified already. If any pair of such Poisson brackets is compatible (see, for instance, [14]), then this pair automatically creates an integrable hydrodynamic chain together with recursion (hereditary) operator.

Acknowledgement

I thank Gennady El, Eugeni Ferapontov, Andrey Maltsev and Sergey Tsarev for their suggestions significantly improving my results.

I am grateful to the Institute of Mathematics in Taipei (Taiwan) where some part of this work has been done, and especially to Jen-Hsu Chang, Jyh-Hao Lee, Ming-Hien Tu and Derchyi Wu for fruitful discussions.

References

[1] *A.A. Balinsky, S.P. Novikov,* Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. (Russian) Dokl. Akad. Nauk SSSR, **283** No. 5 (1985) 1036–1039, translation in Soviet Math. Dokl., **32** (1985) 228-231.

[2] *D.J. Benney*, Some properties of long non-linear waves, Stud. Appl. Math., **52** (1973) 45-50.

[3] *M. Błaszak, B.M. Szablikowski,* Classical $R-$matrix theory of dispersionless systems: I. (1+1)-dimension theory, J. Phys. A: Math. Gen., **35** (2002) 10325-10344. *M. Błaszak, B.M. Szablikowski*, Classical $R-$matrix theory of dispersionless systems: II. (2+1)-dimension theory, J. Phys. A: Math. Gen., **35** (2002) 10345-10364.

[4] *I.Ya. Dorfman*, Dirac structures and integrability of nonlinear evolution equations; Nonlinear Science: Theory and Applications, John Wiley & Sons, New York (1993) 176 pp.

[5] *B.A. Dubrovin, S.P. Novikov,* Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl., **27** (1983) 665–669. *B.A. Dubrovin, S.P. Novikov,* Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, Russian Math. Surveys, **44** No. 6 (1989) 35–124.

[6] *E.V. Ferapontov,* Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, Amer. Math. Soc. Transl. (2), **170** (1995) 33-58.

[7] *E.V. Ferapontov, K.R. Khusnutdinova*, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys., **248** (2004) 187-206, *E.V. Ferapontov, K.R. Khusnutdinova*, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen., **37** No. 8 (2004) 2949 - 2963.

[8] *E.V. Ferapontov, D.G. Marshall*, Differential-geometric approach to the integrability of hydrodynamic chains: the Haanties tensor, *arXiv:nlin.SI/0505013*.

[9] *E.V. Ferapontov, O.I. Mokhov*, Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature, Russian Math. Surveys, **45** No. 3 (1990) 218–219.

[10] *E.V. Ferapontov, M.V. Pavlov*, Reciprocal transformations of Hamiltonian operators of hydrodynamic type: nonlocal Hamiltonian formalism for linearly degenerate systems, J. Math. Phys., **44** No. 3 (2003) 1150-1172.

[11] *N.I. Grinberg*, Poisson brackets of hydrodynamic type with degenerate metric, Russian Math. Surveys, **40** No. 4 (1985) 231-232.

[12] *B.A. Kupershmidt*, Deformations of integrable systems, Proc. Roy. Irish Acad. Sect. A, **83** No. 1 (1983) 45-74. *B.A. Kupershmidt*, Normal and universal forms in integrable hydrodynamical systems, Proceedings of the Berkeley-Ames conference on nonlinear problems in control and fluid dynamics (Berkeley, Calif., 1983), in Lie Groups: Hist., Frontiers and Appl. Ser. B: Systems Inform. Control, II, Math Sci Press, Brookline, MA, (1984) 357-378.

[13] *B.A. Kupershmidt, Yu.I. Manin*, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations, Func. Anal. Appl., **12** No. 1 (1978) 25–37. *D.R. Lebedev, Yu.I. Manin*, Conservation laws and representation of Benney's long wave equations, Phys. Lett. A, **74** No. 3,4 (1979) 154-156.

[14] *F. Magri*, A simple model of the integrable Hamiltonian equation, J. Math. Phys., **19** No.5 (1978) 1156-1162.

[15] *A.Ya. Maltsev, S.P. Novikov*, On the local systems Hamiltonian in the weakly nonlocal Poisson brackets, Physica D, **156** (2001) 53-80.

[16] *M. Manas, L.M. Alonso, E. Medina*, Reductions of the dispersionless KP hierarchy, Theor. Math. Phys., **133** No. 3 (2002) 1712-1721, *B. Konopelchenko, L.M. Alonso, E. Medina*, Quasiconformal mappings and solutions of the dispersionless KP hierarchy, Theor. Math. Phys., **133** No. 2 (2002) 1529-1538.

[17] *O.I. Mokhov*, The Liouville canonical form of compatible nonlocal Poisson brackets of hydrodynamic type, and integrable hierarchies. (Russian) Funktsional. Anal. i Prilozhen., **37** No. 2 (2003) 28–40; translation in Funct. Anal. Appl., **37** No. 2 (2003) 103–113.

[18] *O.I. Mokhov*, Compatible nonlocal Poisson brackets of hydrodynamic type and related integrable hierarchies. (Russian) Teoret. Mat. Fiz., **132** No. 1 (2002) 60–73; translation in Theoret. and Math. Phys., **132** No. 1 (2002) 942–954

[19] *M.V. Pavlov*, Relationships between differential substitutions and Hamiltonian structures of the Korteweg-de Vries equation. Phys. Lett. A, **243** No. 5-6 (1998) 295-300.

[20] *M.V. Pavlov*, Classification of the Egorov hydrodynamic chains, Theor. Math. Phys., **138** No. 1 (2004) 55-71.

[21] *M.V. Pavlov*, The Kupershmidt hydrodynamic chains.

[22] *M.V. Pavlov*, Explicit solutions of the WDVV equation determined by "flat" hydrodynamic reductions of the dispersionless limit for KP, BKP–VN and Boyer–Finley equations.

[23] *V.M. Teshukov*, Characteristics, conservation laws and symmetries of the kinetic equations of motion of bubbles in a fluid, Prikl. Mekh. Tekhn. Fiz., **40** No. 2 (1999) 86-100; translation in J. Appl. Mech. Tech. Phys., **40** No. 2 (1999) 263-275.

[24] *S.P. Tsarev*, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., **31** (1985) 488–491. *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya, **37** No. 2 (1991) 397–419.

[25] *V.E. Zakharov*, Dispersionless limit of integrable systems in 2+1 dimensions, in Singular Limits of Dispersive Waves, Ed. N.M. Ercolani et al., Plenum Press, NY, (1994) 165-174.